Periodic Orbits of Spatial Kepler Problem

Dongho Lee

July 17, 2025

Seoul National University (SNU), Center for Quantum Structures in Modules and Spaces (QSMS)

Presented in Billiards and Quantitative Symplectic Geometry, Heidelberg

Overview

- Based on my paper arXiv:2506.14325.
- Extended result of the 2013 paper "The Conley-Zehnder indices of the rotating Kepler problem" by P. Albers, J. Fish, U. Frauenfelder and O. van Koert. (dim 2 $\to$ dim 3)
- **Result 1**. Description of the moduli space of periodic Kepler orbits using angular momentum and Laplace-Runge-Lenz vector.
- **Result 2**. Computation of Conley-Zehnder indices of periodic Kepler orbits.

Kepler Problem

Rotating Kepler problem is defined by Hamiltonian

$$H = E + L_3 = \frac{1}{2}|p|^2 - \frac{1}{|q|} + (q_1 p_2 - q_2 p_1).$$

H : **Jacobi energy** (usually, $H = c$)

E : **Kepler energy**.

Motivation: a limit of the circular restricted three-body problem

Kepler's Laws

In 17th century, Kepler established these three laws. Let $E < 0$.

1. E-orbits are ellipses with one focus at the origin.
2. The angular momentum is a conserved quantity.
3. The period τ is given by

$$\tau^2 = \frac{\pi^2}{(-2E)^3}$$

Two Invariants

1. **Angular momentum** $L = q \times p$.
   - Direction of $L =$ Normal to the plane which the orbit contained in.
2. **Laplace-Runge-Lenz vector** $A = p \times L - \frac{q}{|q|}$
   - Direction of $A =$ Direction of the major axis

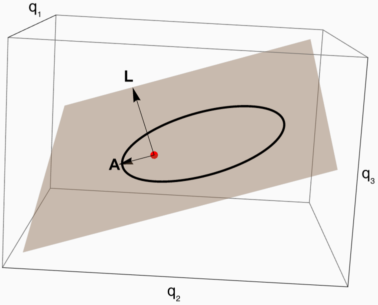
Some relations :

1. $\{E, L_i\} = \{E, A_j\} = 0$ for any i, j.
2. $\{L_i, A_j\} = \varepsilon_{ijk} A_k$. In particular, $\{L_i, A_i\} = 0$
3. Eccentricity: $\varepsilon^2 = |A|^2 = 2E|L|^2 + 1$.

Two Invariants

On $L \cdot q$, an E-orbit is given in the polar coordinate by

$$r = \frac{|L|^2}{1 + |A|\cos(\theta - g)} \quad (g \text{ is determined by the direction of } A).$$



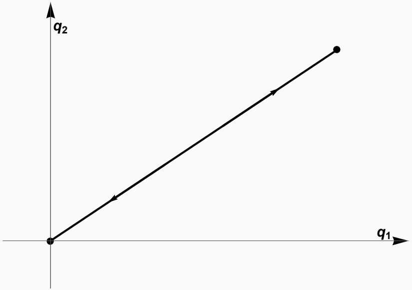
In particular, E, L and A determine the Kepler orbit.

Moser Regularization

For $H < -3/2$, we can embed the Hamiltonian flow on the level set $H^{-1}(c)$ into the unit Finsler geodesic flow on T^*S^3. [CFvK14]

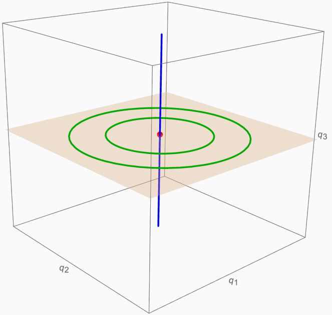
$\Rightarrow$ Compactification of the energy level set by $ST^*S^3 \simeq S^3 \times S^2$.

The **collision orbits** are added. $(\varepsilon = |A| = 1,\ L = 0.)$



Nondegenerate Periodic Orbits

$Fl^{X_{L_3}}$ is a rotation along q_3- and p_3-axis of period 2π, and $Fl^{X_H} = Fl^{X_E} \circ Fl^{X_{L_3}}$.



These are periodic after composing with $Fl^{X_{L_3}}$.

Nondegenerate Periodic Orbits

Circular condition: $\varepsilon^2 = 2EL_3^2 + 1 = 2E(c-E)^2 + 1 = 0$.

For fixed $c < -3/2$, there are 3 **planar circular orbits** with different E.

1. **Retrograde orbit** γ_+: $L_3 > 0$, smaller E and smaller radius.
2. **Direct orbit** γ_-: $L_3 < 0$, larger E and larger radius.
3. The rest one, outer direct orbit, lies on the unbounded component, and not of our interest (discarded during regularization).

Vertical collision orbits $\gamma_{c_\pm}$: $L = 0$, $A_3 = \mp 1$, $c = E$.

- They do not appear in the planar problem.

Morse-Bott Family

For other cases, the periods of E-orbit and L_3-orbit must be the same.

$\tau = 2\pi/(-2E)^{3/2} \Rightarrow$ there exists some $k, l \in \mathbb{Z}$ such that

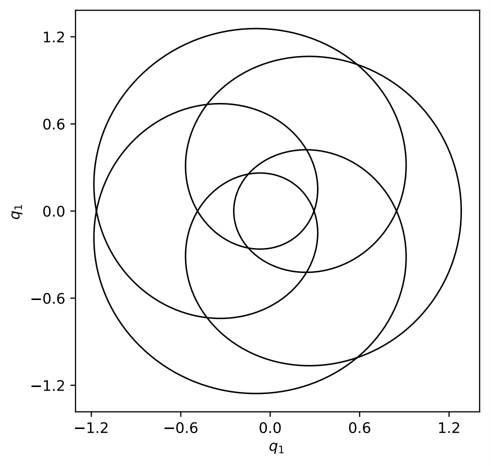
$$k\tau = \frac{2k\pi}{(-2E)^{3/2}} = 2l\pi \;\Rightarrow\; E_{k,l} = -\frac{1}{2}\left(\frac{k}{l}\right)^{2/3}$$

For given c, only orbits with Kepler energy $E_{k,l}$ can be periodic.

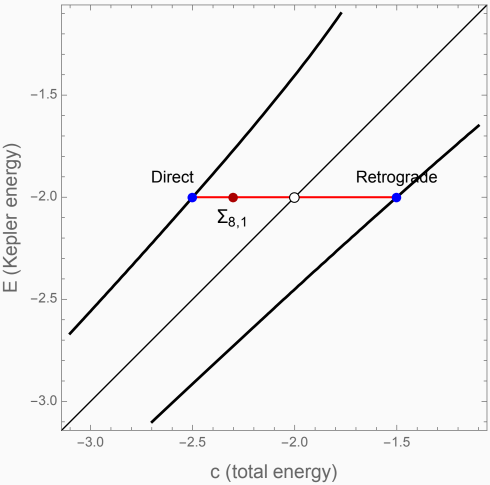
Such orbits appear with Morse-Bott S^3-family $\Sigma_{k,l}$. (will be explained)

Note. We have S^1-families in the planar problem.

Morse-Bott Family



Morse-Bott Family



Moduli Space

Recall. E, L and A characterizes the Kepler orbit.

Denote $x = \sqrt{-2E}L - A$, $y = \sqrt{-2E}L + A$.
$\Rightarrow |x|^2 = |y|^2 = -2E|L|^2 + |A|^2 = 1$.

The moduli space of the Kepler orbits with Kepler energy E is

$$\mathcal{M}_E = \left\{(x, y) : |x|^2 = |y|^2 = 1\right\} \simeq S^2 \times S^2.$$

(Space of unit geodesics of S^3) $= ST^*S^3/S^1 \simeq S^2 \times S^2$.

Note. In the planar problem, the moduli space is $\mathbb{RP}^3/S^1 \simeq S^2$.

Properties of $\mathcal{M}_E$

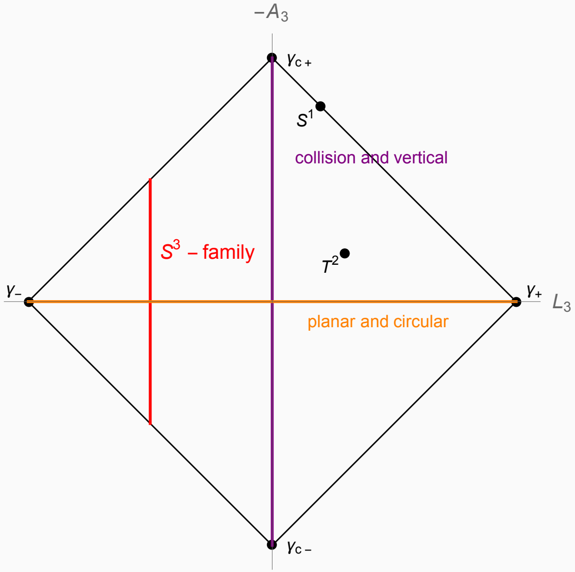
$L_3 = (x_3 + y_3)/\sqrt{-2E}$ serves as a Morse function with 4 critical points.

1. 4 nondegenerate orbits corresponds to the critical points.
   $(0, 0, \pm 1; 0, 0, \pm 1)$
2. Every regular level set of L_3 is S^3. (Handle attachment)

$\Rightarrow$ Morse-Bott family $\Sigma_{k,l}$ is topologically S^3.

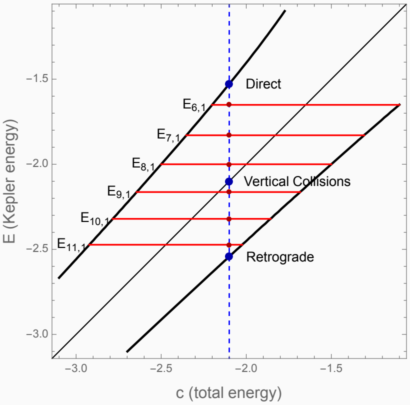
(For fixed c, if $E = E_{k,l}$, then $L_3 = c - E_{k,l}$ is specified.)

Properties of $\mathcal{M}_E$



Periodic Orbits in $H^{-1}(c)$

For generic energy level c, the energy hypersurface $H^{-1}(c)$ contains 4 nondegenerate orbits and (infinitely many) Morse-Bott S^3-families.



Conley-Zehnder Index of Planar Circular Orbits

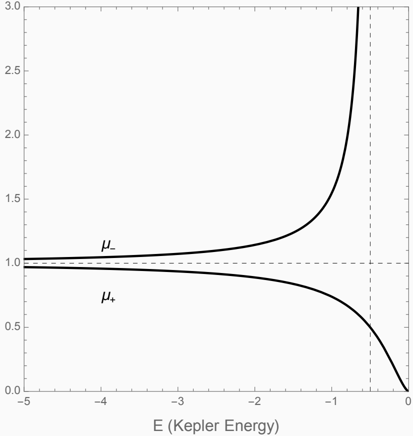
Theorem

Let $\gamma_\pm$ be the retrograde and direct orbits of Kepler energy E where $E \neq E_{k,l}$ for any k, l. Then $\gamma_\pm$ and their multiple covers are non-degenerate. The Conley-Zehnder index of N-th iterate of $\gamma_\pm$ is

$$\begin{aligned}\mu_{CZ}(\gamma_\pm^N) &= 2 + 4\max\left\{n \in \mathbb{Z}_{>0} : n < N\frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}\right\} \\ &= 2 + 4\left\lfloor N\frac{(-2E)^{3/2}}{(-2E)^{3/2} \pm 1}\right\rfloor\end{aligned}$$

The index is exactly the twice compare to the planar problem, which was computed in [AFFvK13].

Conley-Zehnder Index of Planar Circular Orbits



The index of $\gamma_\pm^N$ is initially $4N \mp 2$, and changes by ∓ 4 whenever $\mu_\pm$ touches $k/N \Leftrightarrow E = E_{N\mp k,k}$. (Bifurcation occurs)

Conley-Zehnder Index of Vertical Collision Orbits

Theorem

Let $\gamma_{c_\pm}$ be the vertical collision orbits of Kepler energy E where $E \neq E_{k,l}$ for any k, l. Then $\gamma_{c_\pm}$ and their multiple covers are non-degenerate. The Conley-Zehnder index of N-th iteration of $\gamma_{c_\pm}$ is

$$\mu_{CZ}(\gamma_{c_\pm}^N) = 4N.$$

In particular, change of the energy does not change the index.

Interpretation by Symplectic Homology

$$SH_*^{+,S^1}(T^*S^3;\mathbb{Q}) \simeq \begin{cases} \mathbb{Q} & * = 2, \\ \mathbb{Q}^2 & * = 2k \geq 4, \\ 0 & \text{otherwise.} \end{cases}$$

For fixed N, there exists $c \ll -3/2$ such that $H^{-1}(c)$ consists of

1. $k(\leq N)$-th covers of $\gamma_\pm$ of index $4k \mp 2$. (No bifucation)
2. Higher covers have index $> 4N + 2$.

Up to degree $4N + 2$, we have

1. One generator at degree 2. (γ_+.)
2. Two generators at degree 6, 10, 14, $\cdots$, $4N + 2$. (γ_+^{k+1} and γ_-^k.)
3. Two generators at degree 4, 8, 12, $\cdots$, $4N$. ($\gamma_{c_+}^k$ and $\gamma_{c_-}^k$.)

This describes $SH_*^{+,S^1}(T^*S^3)$ up to degree $4N + 2$ completely.

Morse-Bott Property

$\Sigma_{k,l}$-families: We use **Morse-Bott spectral sequence**.

$\Rightarrow$ We need Morse-Bott property (kind of non-degeneracy), and must compute the linearized return map.

We should use two action-angle coordinates:

1. Delaunay coordinate : $(p_l, p_g, p_\theta) = (1/\sqrt{-2E}, |L|, L_3)$.

   Works for planar problem ([AFFvK13]), but degenerates at every planar orbit in the spatial case.

2. **LRL coordinate** : $(p_l, p_\eta, p_\theta) = (1/\sqrt{-2E}, A_3, L_3)$.

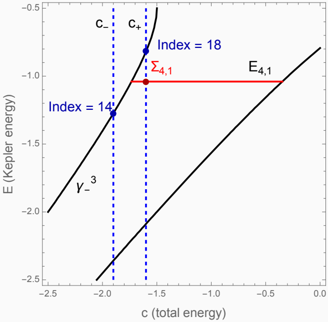
   Also degenerates at some orbits, but covers planar orbits.

Robbin-Salamon Index of Degenerate Orbits

Theorem

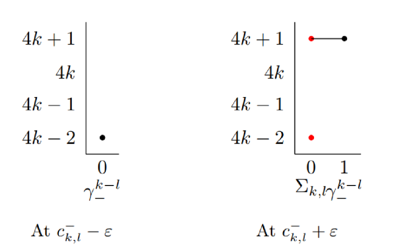
Index of S^3-family $\Sigma_{k,l}$ with Kepler energy $E_{k,l}$ is

$$\begin{aligned}\mu_{RS}(\Sigma_{k,l}) &= \mathrm{shift}(\Sigma) + \dim S^3/2 \\ &= (4k-2) + 3/2 = 4k - 1/2.\end{aligned}$$



Conley-Zehnder Index of Degenerate Orbits

Previous results + local invariance of the symplectic homology



Further Directions

1. Compute the indices for other related problems. (Spatial Euler problem, Hill's lunar problem, etc.)
2. Investigate the bifurcation behavior of other problems.
3. Application to the three-body problem, as a perturbation of the Kepler problem.

Thank you for your attention!